TSP on Cubic and Subcubic Graphs

Boyd, S.M.; Sitters, R.A.; van der Ster, S.L.; Stougie, L.

published in
Lecture Notes in Computer Science
2011

DOI (link to publisher)
10.1007/978-3-642-20807-2_6

document version
Publisher's PDF, also known as Version of record

TSP on Cubic and Subcubic Graphs

Sylvia Boyd[1], René Sitters[2], Suzanne van der Ster[2], and Leen Stougie[2,3,⋆]

[1] School of Information Technology and Engineering (SITE), University of Ottawa, Ottawa, Canada
sylvia@site.uottawa.ca
[2] Department of Operations Research, VU University Amsterdam, The Netherlands
{rsitters,sster,lstougie}@feweb.vu.nl
[3] CWI, Amsterdam, The Netherlands
stougie@cwi.nl

Abstract. We study the Travelling Salesman Problem (TSP) on the metric completion of cubic and subcubic graphs, which is known to be NP-hard. The problem is of interest because of its relation to the famous 4/3 conjecture for metric TSP, which says that the integrality gap, i.e., the worst case ratio between the optimal values of the TSP and its linear programming relaxation, is 4/3. Using polyhedral techniques in an interesting way, we obtain a polynomial-time 4/3-approximation algorithm for this problem on cubic graphs, improving upon Christofides' 3/2-approximation, and upon the $3/2 - 5/389 \approx 1.487$-approximation ratio by Gamarnik, Lewenstein and Svirdenko for the case the graphs are also 3-edge connected. We also prove that, as an upper bound, the 4/3 conjecture is true for this problem on cubic graphs. For subcubic graphs we obtain a polynomial-time 7/5-approximation algorithm and a 7/5 bound on the integrality gap.

1 Introduction

Given a complete undirected graph $G = (V, E)$ on n vertices with non-negative edge costs $c \in \mathbf{R}^E$, $c \neq 0$, the well-known *Traveling Salesman Problem* (TSP) is to find a Hamiltonian cycle in G of minimum cost. When the costs satisfy the triangle inequality, i.e. when $c_{ij} + c_{jk} \geq c_{ik}$ for all $i, j, k \in V$, we call the problem *metric*. A special case of the metric TSP is the so-called *graph-TSP*, where, given an undirected, unweighted simple underlying graph $G = (V, E)$, a complete graph on V is formed, by defining the cost between two vertices as the number of edges on the shortest path between them, known as the *metric completion* of G.

The TSP is well-known to be NP-hard [20], even for the special cases of graph-TSP. As noticed in [17], APX-hardness follows rather straightforwardly from the APX-hardness of (weighted) graphs with edges of length 1 or 2 ((1,2)-TSP) (Papadimitriou and Yannakakis [22]), even if the maximum degree is 6.

⋆ This research was partially supported by Tinbergen Institute, the Netherlands and the Natural Sciences and Engineering Research Council of Canada.

O. Günlük and G.J. Woeginger (Eds.): IPCO 2011, LNCS 6655, pp. 65–77, 2011.

In general, the TSP cannot be approximated in polynomial-time to any constant unless $P = NP$, however for the metric TSP there exists the elegant algorithm due to Christofides [9] from 1976 which gives a 3/2-approximation. Surprisingly, in over three decades no one has found an approximation algorithm which improves upon this bound of 3/2, even for the special case of graph-TSP, and the quest for finding such improvements is one of the most challenging research questions in combinatorial optimization. Very recently, Gharan et al.[16] announced a randomized $3/2 - \epsilon$ approximation for graph-TSP for some $\epsilon > 0$.

A related approach for finding approximated TSP solutions is to study the *integrality gap* $\alpha(TSP)$, which is the worst-case ratio between the optimal solution for the TSP problem and the optimal solution to its linear programming relaxation, the so-called *Subtour Elimination Relaxation* (henceforth SER) (see [5] for more details). The value $\alpha(TSP)$ gives one measure of the quality of the lower bound provided by SER for the TSP. Moreover, a polynomial-time constructive proof for value $\alpha(TSP)$ would provide an $\alpha(TSP)$-approximation algorithm for the TSP.

For metric TSP, it is known that $\alpha(TSP)$ is at most 3/2 (see Shmoys and Williamson [24], Wolsey [25]), and is at least 4/3 (a ratio of 4/3 is reached asymptotically by the family of graph-TSP problems consisting of two vertices joined by three paths of length k; see also [5] for a similar family of graphs giving this ratio), but the exact value of $\alpha(TSP)$ is not known. However, there is the following well-known conjecture:

Conjecture 1. For the metric TSP, the integrality gap $\alpha(TSP)$ for SER is 4/3.

As with the quest to improve upon Christofides' algorithm, the quest to prove or disprove this conjecture has been open for almost 30 years, with very little progress made.

A graph is *cubic* if all of its vertices have degree 3, and *subcubic* if they have degree at most 3. A graph is *k-edge connected* if removal of less than k edges keeps the graph connected. A *bridge* in a connected graph is an edge whose removal breaks the graph into two disconnected subgraphs.

In this paper we study the graph-TSP problem on cubic and subcubic graphs. Note that the graphs in the family described above giving a worst-case ratio of 4/3 for $\alpha(TSP)$ are graph-TSPs on bridgeless subcubic graphs. Our main result improves upon Christofides' algorithm by providing a 4/3-approximation algorithm as well as proving 4/3 as an upper bound in Conjecture 1 for the the special case of graph-TSP for which the underlying graph $G = (V, E)$ is a cubic graph. Note that solving the graph-TSP on such graphs would solve the problem of deciding whether a given bridgeless cubic graph G has a Hamilton cycle, which is known to be NP-complete, even if G is also planar (Garey *et al.* [15]) or bipartite (Akiyama *et al.* [2]). In [8] there is an unproven claim that (1,2)-TSP is APX-hard when the graph of edges of length 1 is cubic, which would imply APX-hardness of graph-TSP on cubic graphs.

Also note that the 3/2 ratio of Christofides' algorithm is tight for cubic graph-TSP (see Figure 1). As noted by Gamarnik *et al.* in [14], one approach that can

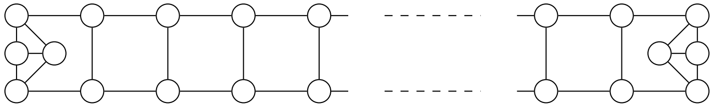

Fig. 1. Example of a cubic graph on which Christofides may attain a ratio of 3/2

be taken for graph-TSP is to look for a polynomial-time algorithm that finds a Hamilton cycle of cost at most τn for some $\tau < 3/2$. Since n is a lower bound for the optimal value for graph-TSP as well as the associated SER[1], this will improve upon Christofides' algorithm by giving a τ-approximation for the graph-TSP, as well as prove that the integrality gap $\alpha(TSP)$ is at most τ for such problems. In [14], Gamarnik *et al.* note a connection between optimal solutions to SER and 3-edge connected cubic graphs. Furthermore, they give an algorithm for graph-TSP where the underlying graph is 3-edge connected and cubic, and for which $\tau = (3/2 - 5/389) \approx 1.487$.

The algorithm of Gamarnik *et al.* provided the first approximation improvement over Christofides' algorithm for the graph-TSP for 3-edge connected cubic graphs. We improve upon their results, both in terms of the value of τ and the class of underlying graphs by proving the following:

Theorem 1. *Every bridgeless simple cubic graph $G = (V, E)$ with $n \geq 6$ has a graph-TSP tour of length at most $\frac{4}{3}n - 2$.*

Our proof of this theorem is constructive, and provides a polynomial-time 4/3-approximation algorithm for graph-TSP on bridgeless cubic graphs. The proof uses polyhedral techniques in a surprising way, which may be more widely applicable. The result also proves that Conjecture 1 is true for this class of TSP problems as an upper bound. The theorem is indeed central in the sense that the other results in this paper are based upon it. One of them is that we show how to incorporate bridges with the same guarantees.

For subcubic graphs it appears to be harder to obtain the same strong results as for cubic graphs. For this class of graph-TSP we obtain a 7/5-approximation algorithm and prove that the integrality gap is bounded by 7/5, still improving considerably over the existing 3/2 bounds. Note that 4/3 is a lower bound for $\alpha(TSP)$ on subcubic graphs.

Relevant literature: Between the first and final submission of this conference paper Aggarwal et al. [1] announced an alternative 4/3 approximation for 3-edge connected cubic graphs. Grigni *et al.* [17] give a polynomial-time approximation scheme (PTAS) for graph-TSP on planar graphs (this was later extended to a PTAS for the weighted planar graph-TSP by Arora *et al.* [3]). For graph G containing a cycle cover with no triangles, Fotakis and Spirakis [12] show that

[1] To see that n is a lower bound for SER, sum all of the so-called "degree constraints" for SER. Dividing the result by 2 shows that the sum of the edge variables in any feasible SER solution equals n.

graph-TSP is approximable in polynomial time within a factor of $17/12 \approx 1.417$ if G has diameter 4 (i.e. the longest path has length 4), and within $7/5 = 1.4$ if G has diameter 3. For graphs that do not contain a triangle-free cycle cover they show that if G has diameter 3, then it is approximable in polynomial time within a factor of $22/15 \approx 1.467$. For graphs with diameter 2 (i.e. TSP(1,2)), a $7/6 \approx 1.167$-approximation for graph-TSP was achieved by Papadimitriou and Yannakakis [22], and improved to $8/7 \approx 1.143$ by Berman and Karpinski [6].

2 Preliminaries

We begin this section with some definitions. Given a graph $G = (V, E)$, we let $V(G)$ denote the vertex set V of G. For any vertex subset $S \subseteq V$, $\delta(S) \subseteq E$, defined as the set of edges connecting S and $V \setminus S$, is called the *cut* induced by S. A cut of cardinality k is called a *k-cut* if it is minimal in the sense that it does not contain any cut as a proper subset.

A *cycle* in a graph is a closed path. In this paper, cycles have no repetition of vertices, which in graph literature is often referred to as *circuits*. A *k-cycle* is a cycle containing k edges, and a *chord* of a cycle of G is an edge not in the cycle, but with both ends u and v in the cycle. A *cycle cover* (also sometimes referred to as 2-factor or perfect 2-matching) of G is a set of vertex disjoint cycles that together span all vertices of G. An *Eulerian subgraph* of G is a connected subgraph where multiple copies of the edges are allowed, and all vertices have even degree. A *perfect matching* M of a graph G is a set of vertex-disjoint edges of G that together span all vertices of G. We call M a 3-*cut perfect matching* if every 3-cut of G contains exactly one edge of M.

A well-known theorem of Petersen [23] states that every bridgeless cubic graph contains a perfect matching. Thus the edges of any bridgeless cubic graph can be partitioned into a perfect matching and an associated cycle cover. This idea is important for our main theorem, and we give a useful strengthened form of it below in Lemma 1.

For any edge set $F \subseteq E$, the *incidence vector of* F is the vector $\chi^F \in \mathbf{R}^E$ defined by $\chi^F_e = 1$ if $e \in F$, and 0 otherwise. For any edge set $F \subseteq E$ and $x \in \mathbf{R}^E$, let $x(F)$ denote the sum $\sum_{e \in F} x_e$.

Given graph G, the associated *perfect matching polytope*, $P^M(G)$, is the convex hull of all incidence vectors of the perfect matchings of G, which Edmonds [11] shows to be given by:

$$
\begin{aligned}
x(\delta(v)) &= 1, \qquad \forall v \in V, \\
x(\delta(S)) &\geq 1, \qquad \forall S \subset V,\ |S| \text{ odd}, \\
0 \leq x_e &\leq 1, \qquad \forall e \in E.
\end{aligned}
$$

Using this linear description and similar methods to those found in [19] and [21], we have the following strengthened form of Petersen's Theorem:

Lemma 1. *Let $G = (V, E)$ be a bridgeless cubic graph and let $x^* = \frac{1}{3}\chi^E$. Then x^* can be expressed as a convex combination of incidence vectors of* 3*-cut perfect matchings, i.e. there exists* 3*-cut perfect matchings M_i, $i = 1, 2, ..., k$ of G and positive real numbers λ_i, $i = 1, 2, ..., k$ such that*

$$x^* = \sum_{i=1}^{k} \lambda_i(\chi^{M_i}) \text{ and } \sum_{i=1}^{k} \lambda_i = 1. \tag{1}$$

Proof. Since both sides of any 2-cut in a cubic graph have an even number of vertices, it is easily verified that x^* satisfies the linear description above, and thus lies in $P^M(G)$. It follows that x^* can be expressed as a convex combination of perfect matchings of G, i.e. there exist perfect matchings M_i, $i = 1, 2, ..., k$ of G and positive real numbers λ_i, $i = 1, 2, ..., k$ such that (1) holds.

To see that each perfect matching in (1) is a 3-cut perfect matching, consider any 3-cut $\delta(S) = \{e_1, e_2, e_3\}$ of G. Since each side of a 3-cut of any cubic graph must contain an odd number of vertices, any perfect matching must contain 1 or 3 edges of $\delta(S)$. Let $\mathcal{M}_0$ be the set of perfect matchings from (1) that contain all 3 edges of the cut, and let $\mathcal{M}_j$, $j = 1, 2, 3$ be the sets of perfect matchings that contain edge e_j. Define $\alpha_j = \sum_{M_i \in \mathcal{M}_j} \lambda_i$, $j = 0, 1, 2, 3$. Then

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = x^*(\delta(S)) = 1$$
$$\alpha_0 + \alpha_1 = 1/3, \; \alpha_0 + \alpha_2 = 1/3, \; \alpha_0 + \alpha_3 = 1/3,$$

which implies $\alpha_0 = 0$. □

The perfect matchings $M_i, i = 1, 2, ...k$ of Lemma 1 will be used in the proof of our main theorem in the next section. Note that Barahona [4] provides an algorithm to find for any point in $P^M(G)$ a set of perfect matchings for expressing the point as a convex combination of their incidence vectors in $O(n^6)$ time, and with $k \leq 7n/2 - 1$, for any graph G.

3 Cubic Graphs

In our analysis of the graph-TSP problem for graph $G = (V, E)$, we will consider the equivalent form of the problem, introduced in [10] as the *graphical TSP* of G (henceforth GTSP), in which one seeks a minimum length tour of G in which vertices can be visited more than once and edges can be traversed more than once. The solution, which we refer to as a *GTSP tour*, forms a spanning Eulerian subgraph $H = (V, E')$ of G, which can be transformed into a graph-TSP tour of G of cost $|E'|$ and vice versa. Note that an edge appears at most twice in H.

The idea we will use in the proof of our main theorem is as follows: By Petersen's Theorem we know we can always find a cycle cover of G. Suppose that we can find such a cycle cover that has no more than $n/6$ cycles. Then, contracting the cycles, adding a doubled spanning tree in the resulting graph and uncontracting the cycles would yield a GTSP solution with no more than

$n + 2(n/6 - 1) = 4n/3 - 2$ edges. Together with the obvious lower bound of n on the length of any optimal GTSP tour, this yields an approximation ratio of $4/3$. However, such a cycle cover does not always exist (for example, consider the Petersen graph)[2]. Therefore, we take the k cycle covers associated with the 3-cut matchings of Lemma 1 and combine their smaller cycles into larger cycles or Eulerian subgraphs, such as to obtain k covers of G with Eulerian subgraphs which, together with the double spanning tree, result in k GSTP tours with average length at most $4/3n$. For this construction of larger Eulerian subgraphs the following lemma will be useful.

Lemma 2. *Let H_1 and H_2 be connected Eulerian subgraphs of a (sub)cubic graph such that H_1 and H_2 have at least two vertices in common and let H_3 be the* sum *of H_1 and H_2, i.e., the union of their vertices and the sum of their edges, possibly giving rise to parallel edges. Then we can remove two edges from H_3 such that it stays connected and Eulerian.*

Proof. Let u and v be in both subgraphs. The edge set of H_3 can be partitioned into edge-disjoint (u, v)-walks P_1, P_2, P_3, P_4. Vertex u must have two parallel edges which are on different paths, say $e_1 \in P_1$ and $e_2 \in P_2$. When we remove e_1 and e_2 then the graph stays Eulerian. Moreover, it stays connected since u and v are still connected by P_3 and P_4 and, clearly, each vertex on P_1 and P_2 remains connected to either u or v. □

The following lemma, which applies to any graph, allows us to preprocess our graph by removing certain subgraphs.

Lemma 3. *Assume that removing edges $u'u''$ and $v'v''$ from graph $G = (V, E)$ breaks it into two graphs $G' = (V', E')$ and $G'' = (V'', E'')$ with $u', v' \in V'$, and $u'', v'' \in V''$ and such that:*

1. *$u'v' \in E$ and $u'', v'' \notin E$.*
2. *there is a GTSP tour T' in G' of length at most $4|V'|/3 - 2$.*
3. *there is a GTSP tour T'' in $G'' \cup u''v''$ of length at most $4|V''|/3 - 2$.*

Then there is a GTSP tour T in G of length at most $4|V|/3 - 2$.

Proof. If T'' does not use edge $u''v''$ then we take edge $u'u''$ doubled and add tour T'. If T'' uses edge $u''v''$ once then we remove it and add edges $u'u''$, $v'v''$ and $u'v'$ and tour T'. If T'' uses edge $u''v''$ twice then we remove both copies and add edge $u'u''$ doubled, $v'v''$ doubled, and tour T'. □

We use Lemma 3 to remove all subgraphs of the form shown in Figure 2, which we call a *p-rainbow* subgraph. In such subgraphs there is a path $u_0, u_1, \ldots, u_{p+1}$ and path $v_0, v_1, \ldots, v_{p+1}$ for some $p \geq 1$, and a 4-cycle u_0, a, v_0, b with chord ab. Furthermore, there are edges $u_i v_i$ for each $i \in \{1, 2, \ldots, p\}$ but there is no edge

[2] We remark that if G is 3-edge connected and cubic there does exist a triangle- and square-free cycle cover of G which can be found in polynomial time (see [7],[18]), resulting in a straightforward 7/5 approximation algorithm for such graphs.

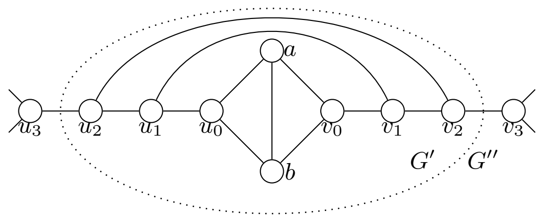

Fig. 2. In this p-rainbow example, $p = 2$ and $u' = u_2, u'' = u_3, v' = v_2$, and $v'' = v_3$

between u_{p+1} and v_{p+1}. The figure shows a p-rainbow for $p = 2$. For general p, the 2-cut of Lemma 3 is given by $u' = u_p$, $u'' = u_{p+1}$, $v' = v_p$, and $v'' = v_{p+1}$. If G contains a p-rainbow G', $p \geq 1$, then we remove G' and add edge $u''v''$ to the remaining graph G''. Note that G'' is also a simple bridgeless cubic graph. We repeat this until there are no more p-rainbows in G'' for any $p \geq 1$. If the final remaining graph G'' has at least 6 vertices, then assuming G'' has a GTSP tour of length at most $4/3|V''| - 2$, we can apply Lemma 3 repeatedly to obtain a GTSP tour of length at most $4/3n - 2$ for the original graph G. If the final remaining graph G'' has less than 6 vertices, then it must have 4 vertices, since it is cubic, hence it forms a complete graph on 4 vertices. In this case we take the Hamilton path from u'' to v'' in G'' and match it with the Hamilton path of the p-rainbow that goes from u_p to v_p to obtain a Hamilton cycle of the graph G'' with the edge $u''v''$ replaced by the p-rainbow. We can then apply Lemma 3 repeatedly to obtain a GTSP tour of length at most $4/3n - 2$ for G.

Proof of Theorem 1. By the above discussion, we assume that there are no p-rainbow subgraphs in G. By Lemma 1 there exist 3-cut perfect matchings $M_1, \ldots, M_k$ and positive real numbers $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\frac{1}{3}\chi^E = \sum_{i=1}^{k} \lambda_i(\chi^{M_i})$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the cycle covers of G corresponding to $M_1, M_2, \ldots M_k$. Since each M_i is a 3-cut perfect matching, each $\mathcal{C}_i$ intersects each 3-cut of G in exactly 2 edges, and hence contains neither a 3-cycle nor a 5-cycle with a chord.

If some $\mathcal{C}_i$ has no more than $n/6$ cycles, then we are done, by the argument given earlier. Otherwise we manipulate each of the cycle covers by operations (i) and (ii) below, which we will show to be well-defined. First operation (i) will be performed as long as possible. Then operation (ii) will be performed as long as possible.

(i) If two cycles C_i and C_j of the cycle cover intersect a (chordless) cycle C of length 4 in G (the original graph) then combine them into a single cycle on $V(C_i) \cup V(C_j)$.

The details of operation (i) are as follows: Assume that u_1u_2 and v_1v_2 are edges of C (and the matching) such that u_1v_1 is an edge of C_i and u_2v_2 is an edge of C_j. Deleting the latter two edges and inserting the former two yields a single cycle of length equal to the sum of the lengths of C_i and C_j. Notice that operation (i) always leads to cycles of length at least 8. Hence after operation (i) is finished we still have a cycle cover. Operation (ii) below combines cycles into Eulerian subgraphs and subsequently Eulerian subgraphs into larger Eulerian subgraphs, turning the cycle covers into Eulerian subgraph covers. Both types of cover we call simply a *cover* and their elements (cycles and Eulerian subgraphs) we call *components.*

(ii) If two components γ_i and γ_j of the cycle cover or the Eulerian subgraph cover, each having at least 5 vertices, intersect a (chordless) cycle C of length 5 in G (the original graph) then combine them into a single Eulerian subgraph where the number of edges is 1 plus the number of edges of γ_i and γ_j.

The details of operation (ii) are as follows. First note that for any cycle C, its vertex set $V(C)$ has the following (trivial) property:

$\mathcal{P}$: Each $v \in V(C)$ has at least two other vertices $u, w \in V(C)$ such that $vu \in E$ and $vw \in E$.

If two vertex sets both satisfy $\mathcal{P}$ then their union also satisfies $\mathcal{P}$. Since the vertex set of each component γ constructed by operations (i) or (ii) is a result of taking unions of vertex sets of cycles, each such γ has property $\mathcal{P}$. In particular, since G is cubic, this implies that the two components γ_i and γ_j share 2 and 3 vertices with C, respectively (note that they cannot each share exactly 2 vertices, as this would imply that a vertex of C is not included in the cover). We first merge γ_1 and C as in Lemma 2 and remove 2 edges, and then merge the result with γ_2, again removing 2 edges. Altogether we added the 5 edges of C and removed 4 edges.

Operation (ii) leads to Eulerian subgraphs with at least 10 vertices. Thus, any Eulerian subgraph with at most 9 vertices is a cycle. At the completion of operations (i) and (ii), let the resulting Eulerian subgraph covers be $\Gamma_1, \ldots, \Gamma_k$.

Given $\Gamma_1, \ldots, \Gamma_k$, we bound for each vertex its average contribution to the cost of the GTSP tours, weighted by the λ_i's. We define the contribution of a vertex v which in cover Γ_i lies on an Eulerian subgraph with ℓ edges and h vertices as $z_i(v) = \frac{\ell+2}{h}$; the 2 in the numerator is added for the cost of the double edge to connect the component to the others in the GTSP tour. Note that $\sum_{v \in V} z_i(v)$ is equal to the length of the GTSP solution corresponding to Γ_i, plus 2. The average contribution of v over all covers is $z(v) = \sum_i \lambda_i z_i(v)$. When summing this over all vertices v we obtain the average length of the GTSP tours plus 2. We will show that $z(v) \leq 4/3 \ \forall v \in V$.

Observation 1. *For any vertex v and $i \in \{1, 2, \ldots, k\}$, the contribution $z_i(v)$ is*

(a) at most $\frac{h+2}{h}$, where $h = \min\{t, 10\}$ and v is on a cycle of length t in $\mathcal{C}_i$ or after operation (i).
(b) at most $13/10$ if operation (ii) was applied to some component containing v.

Proof (Observation 1). Assume that v is on a Eulerian subgraph γ in Γ_i of g vertices. First we prove (b). If operation (ii) was applied to some component containing v, then vertex v was on a cycle of length at least 5 after operation (i). Each application of (ii) adds at least 5 vertices to the component of v. Hence, the number of times that (ii) was applied to the component of v is at most $g/5-1$. Since each application adds exactly one edge, the number of edges in γ is at most $g+g/5-1$. Hence,

$$z_i(v) \leq \frac{g+g/5+1}{g} = \frac{12}{10} + \frac{1}{g} \leq \frac{13}{10}.$$

We use a similar argument to prove (a). Clearly, $g \geq h$. If γ is a cycle then the contribution of v in Γ_i is $(g+2)/g \leq (h+2)/h$ and (a) is true. If γ is not a cycle then this Eulerian subgraph was composed by operation (ii) applied to cycles, each of length at least 5 and one of these had length at least h. Hence, the number of these cycles is at most $1+(g-h)/5$. Since every application of operation (ii) adds one edge extra, the number of edges in γ is at most $g+(g-h)/5$. Hence, since $h \leq 10$,

$$z_i(v) \leq \frac{g+(g-h)/5+2}{g} \leq \frac{g+(g-h)/(h/2)+2}{g} = \frac{h+2}{h}. \qquad \square$$

Note the subtleties in Observation 1: If v is on a cycle of length t in $\mathcal{C}_i$ or after operation (i), and $t \leq 10$, then (a) says that $z_i(v)$ is at most $(t+2)/t$. if $t > 10$, then (a) says that its contribution is at most 12/10. And finally, if t is 5 or 6 and we know that operation (ii) was applied to some component containing v, then (b) allows us to improve the upper bound on $z_i(v)$ to 13/10 (for other values of t, (b) does not give an improvement).

From now on we fix any vertex v. Suppose that there is no ℓ such that v is on a 4-cycle or a 5-cycle of Γ_ℓ. Then using Observation 1, we have $z_i(v) \leq \max\{8/6, 13/10\} = 4/3$ for every cover Γ_i, and thus $z(v) \leq 4/3$ and we are done.

Now suppose there exists an ℓ such that v is on a 4-cycle C of Γ_ℓ. Then C must be present in $\mathcal{C}_\ell$ as well. First assume that C is chordless in G. Then all four edges adjacent to C are in the set M_ℓ.

Observation 2. *For any pair of vertices on a chordless cycle of* G *that appears in any* $\mathcal{C}_i$*, any path between the two that does not intersect the cycle has length at least* 3.

We partition the set $\mathcal{C}_1, \ldots, \mathcal{C}_k$ according to the way the corresponding M_i's intersect the cycle C. Define sets X_0, X_1, X_2 where $X_j = \{i \mid |C \cap M_i| = j\}$ for $j = 0,1,2$. Let $x_t = \sum_{i \in X_t} \lambda_i$, $t = 0,1,2$. Clearly $x_0 + x_1 + x_2 = 1$. Since each of the four edges adjacent to C receives total weight 1/3 in the matchings, we have that $4x_0 + 2x_1 = 4/3 \Rightarrow x_0 = 1/3 - x_1/2$. Since each of the edges of C receives total weight 1/3 in the matchings, $x_1 + 2x_2 = 4/3 \Rightarrow x_2 = 2/3 - x_1/2$.

Clearly, for any $i \in X_0$, v lies on cycle C in $\mathcal{C}_i$, and thus by Observation 1(a), $z_i(v) \leq 6/4$. By Observation 2, for any $i \in X_1$, v lies on a cycle of length at least 6 in $\mathcal{C}_i$, and thus by Observation 1(a), $z_i(v) \leq 8/6$. For any $i \in X_2$, if C is intersected by one cycle in $\mathcal{C}_i$, then this cycle has length at least 8 by Observation 2. If for $i \in X_2$, C is intersected by two cycles of length at least 4 each, then, after performing operation (i), v will be on a cycle of length at least 8. Thus using Observation 1(a) one more time, we obtain

$$\begin{aligned} z(v) &\leq x_0 6/4 + x_1 8/6 + x_2 10/8 \\ &= (1/3 - x_1/2)6/4 + x_1 8/6 + (2/3 - x_1/2)10/8 \\ &= 4/3 + x_1(8/6 - 6/8 - 10/16) = 4/3 - x_1/24 \leq 4/3. \end{aligned}$$

We prove now that $z(v) \leq 4/3$ also if C is a 4-cycle with a chord. Let us call the vertices on the cycle u_0, a, v_0, b, let ab be the chord, and v is any of the four vertices. If $u_0v_0 \in E$, then $G = K_4$ (the complete graph on 4 vertices), contradicting the assumption that $n \geq 6$. Thus edges u_0u_1 and v_0v_1 exist, with $u_1, v_1 \notin C$. Notice that $u_1 \neq v_1$ since otherwise G would contain a bridge, contradicting 2-connectedness. Let C' be the cycle containing v in some cycle cover $\mathcal{C}_i$. If C' does not contain edge u_0u_1 then $C' = C$. If, on the other hand, $u_0u_1 \in C'$ then also $v_0v_1 \in C'$ and $ab \in C'$. Note that $u_1v_1 \notin E$ since otherwise we have a p-rainbow subgraph as in Figure 2, and we are assuming that we do not have any such subgraphs. Consequently, C' cannot have length exactly 6. It also cannot have length 7 since then a 3-cut with 3 matching edges would occur. Therefore, any cycle containing u_0u_1 has length at least 8. Applying Observation 1(a) twice we conclude that $z(v) \leq 1/3 \cdot 6/4 + 2/3 \cdot 10/8 = 4/3$.

Now assume there exists a (chordless) 5-cycle C containing v in some Γ_ℓ. Note that we can assume that no $w \in C$ is on a 4-cycle of G, otherwise operation (i) would have been applied and the component of v in Γ_ℓ would have size larger than 5. Note further that C is present in $\mathcal{C}_\ell$ as well. The proof for this case is rather similar to the case for the chordless 4-cycle. Let X_j be the set $\{i \mid |C \cap M_i| = j\}$, for $j = 0, 1, 2$. Let $x_t = \sum_{i \in X_t} \lambda_i$, $t = 0, 1, 2$. Again, we have $x_0 + x_1 + x_2 = 1$. Clearly, for any $i \in X_0$, v lies on C in $\mathcal{C}_i$ and for $i \in X_1$ v lies on a cycle of length at least 7 by Observation 2. Hence, by Observation 1(a) we have $z_i(v) \leq 7/5$ for $i \in X_0$ and $z_i(v) \leq 9/7$ for $i \in X_1$. For any $i \in X_2$ there are two possibilities: Either C is intersected by one cycle in $\mathcal{C}_i$, which, by Observation 2, has length at least 9, or C is intersected in $\mathcal{C}_i$ by two cycles, say C_1 and C_2. In the first case we have $z_i(v) \leq 11/9$ by Observation 1(a). In the second case, as argued before, we can assume that no $w \in C$ is on a 4-cycle of G. Hence, C_1 and C_2 each have at least 5 vertices and operation (ii) will be applied, unless C_1 and C_2 end up in one large cycle by operation (i). In the first case we apply Observation 1(b) and get $z_i(v) \leq 13/10$, and in the second case we apply Observation 1(a): $z_i(v) \leq 12/10$. Hence, for any $i \in X_2$ we have $z_i(v) \leq \max\{11/9, 12/10, 13/10\} = 13/10$.

$$\begin{aligned} z(v) &\leq x_0 7/5 + x_1 9/7 + x_2 13/10 \\ &\leq x_0 7/5 + x_1 13/10 + x_2 13/10 \\ &= x_0 7/5 + (1 - x_0)13/10 = 13/10 + x_0 1/10 \\ &\leq 13/10 + 1/30 = 4/3. \end{aligned}$$

□

As previously mentioned, Barahona [4] provides a polynomial-time algorithm which finds a set of at most $7n/2 - 1$ perfect matchings such that $\frac{1}{3}\chi^E$ can be expressed as a convex combination of the incidence vectors of these matchings. This algorithm runs in $O(n^6)$ time. As shown in the proof of Lemma 1, these matchings will automatically be 3-cut perfect matchings. Once we have this set of perfect matchings then applying operations (i) and (ii) on the corresponding cycle covers gives at least one tour of length at most $4n/3 - 2$ according to the above theorem. As any tour has length at least n for graph-TSP, we have the following approximation result:

Corollary 1. *For graph-TSP on bridgeless cubic graphs there exist a polynomial-time* $4/3$ *approximation algorithm.*

As n is a lower bound on the value of SER for graph-TSP it also follows that, as an upper bound, Conjecture 1 is true for this class of problems, i.e.,

Corollary 2. *For graph-TSP on bridgeless cubic graphs the integrality gap for SER is at most* $4/3$.

We remark that the largest ratio we found so far for $\alpha(TSP)$ on bridgeless cubic examples is 7/6. Without proof we extend the result to include bridges.

Theorem 2. *For a cubic graph with b bridges and s vertices incident to more than one bridge, a TSP tour of length at most* $(4/3)(n + b - s) - 2$ *can be constructed in polynomial time.*

Since an optimal tour on a graph with b bridges has at least $n + 2b - s$ edges:

Corollary 3. *For graph-TSP on cubic graphs, there exists a polynomial-time* $4/3$*-approximation algorithm, and the integrality gap for SER is at most* $4/3$.

4 Subcubic Graphs

When we allow vertices of degree 2, i.e., we consider 2-connected graphs of maximum degree 3, then the optimal GTSP tour may be as large as $4n/3 - 2/3$. For example, take two vertices joined by three paths of the same length. We conjecture that this bound is tight but have no proof. Instead we can show a bound of $7n/5 - 4/5$, based on relating the cubic graph result to this case. Proofs are omitted from this section.

Theorem 3. *Every* 2*-edge connected graph of maximum degree* 3 *has a TSP tour of length at most* $\frac{7}{5}n - \frac{4}{5}$.

As with the cubic case, this result can be extended to include bridges.

Theorem 4. *For a graph of maximum degree* 3 *consisting of n vertices and b bridges, a TSP tour of length at most* $7(n - s + 2b)/5$ *can be constructed.*

From the proofs of Theorems 3 and 4 a polynomial-time algorithm can be designed. Since $n + 2b - s$ can be shown to be a lower bound for graph-TSP and for SER on subcubic graphs with b bridges, we have:

Corollary 4. *For graph-TSP on subcubic graphs, there exists a polynomial-time* 7/5*-approximation algorithm, and the integrality gap for SER is at most* 7/5*.*

5 Epilogue

The table below shows the state of knowledge about graph-TSP for various classes of graphs. It contains: (column A) lower bounds on the length of graph-TSP tours on n vertices, for n large enough, (column B) upper bounds on them that we know how to construct, (column C) lower bounds on the integrality gap of SER, (column D) upper bounds on the integrality gap of SER, and (column E) upper bounds on the best possible approximation ratio. We have selected only the bridgeless cases, because they are the crucial ones within the classes. Columns B,D and E in the last two rows comprises our work. The other results, specifically the lower bounds in the last 2 rows are briefly explained in the full version of this paper.

		A-lower	B-upper	C-lb-sep	D-ub-sep	E-apr
1	general, 2-edge connected	$2n-4$	$2n-2$	$4/3$	$3/2$	$3/2$
2	max degree 3, 2 edge conn	$4n/3-2/3$	$7n/5-4/5$	$4/3$	$7/5$	$7/5$
3	cubic 2 edge connected	$11n/9-10/9$	$4n/3-2$	$7/6$	$4/3$	$4/3$

The table shows a gap in our knowledge for each of the problem classes. Closing these gaps is an obvious open research question. Proving APX-hardness of cubic graph-TSP is open as well. Another interesting research question is to improve on the running time of our algorithm, which is highly dominated by the $O(n^6)$-time algorithm of Baharona which works for every graph and every point in the perfect matching polytope. Can we find a faster algorithm for the special case that the graph is cubic and for the special point $\frac{1}{3}\chi^E$? The question is related to the Berge-Fulkerson Conjecture [13] which implies that the point $\frac{1}{3}\chi^E$ can be expressed as the convex combination of at most 6 perfect matchings.

Of course, the main research challenges remain to prove Conjecture 1 or even show a 4/3-approximation algorithm. Also for the special case of graph-TSP this problem is wide open.

References

1. Aggarwal, N., Garg, N., Gupta, S.: A 4/3-approximation for TSP on cubic 3-edge-connected graphs (2011) (manuscript)
2. Akiyama, T., Nishizeki, T., Saito, N.: NP-completeness of the hamiltonian cycle problem for bipartite graphs. Journal of Information Processing 3, 73–76 (1980)
3. Arora, S., Grigni, M., Karger, D., Klein, P., Woloszyn, A.: A polynomial-time approximation scheme for weighted planar graph TSP. In: Proc. of the 9th ACM–SIAM Symposium on Discrete Algorithms, pp. 33–41 (1998)
4. Barahona, F.: Fractional packing of T-joins. SIAM Journal on Discrete Math. 17, 661–669 (2004)

5. Benoit, G., Boyd, S.: Finding the exact integrality gap for small travelling salesman problems. Math. of Operations Research 33, 921–931 (2008)
6. Berman, P., Karpinski, M.: 8/7-approximation algorithm for 1,2-TSP. In: Proc. 17th ACM SIAM Symposium on Discrete Algorithms, pp. 641–648 (2006)
7. Boyd, S., Iwata, S., Takazawa, K.: Finding 2-Factors Covering 3- and 4-Edge Cuts in Bridgeless Cubic Graphs Kyoto University (2010) (manuscript)
8. Csaba, B., Karpinski, M., Krysta, P.: Approximability of dense and sparse instances of minimum 2-connectivity, tsp and path problems. In: Proc. 13th ACM–SIAM Symposium on Discrete Algorithms, pp. 74–83 (2002)
9. Christofides, N.: Worst case analysis of a new heuristic for the traveling salesman problem, Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh (1976)
10. Cornuéjols, G., Fonlupt, J., Naddef, D.: The traveling salesman on a graph and some related integer polyhedra. Math. Programming 33, 1–27 (1985)
11. Edmonds, J.: Maximum matching and a polyhedron with 0,1-vertices. J. of Res. National Bureau of Standards B 69, 125–130 (1965)
12. Fotakis, D., Spirakis, P.: Graph properties that facilitate travelling. Electronic Colloquium on Computational Complexity 31, 1–18 (1998)
13. Fulkerson, D.: Blocking and anti-blocking pairs of polyhedra. Math Programming 1, 168–194 (1971)
14. Gamarnik, D., Lewenstein, M., Sviridenko, M.: An improved upper bound for the TSP in cubic 3-edge-connected graphs. OR Letters 33, 467–474 (2005)
15. Garey, M., Johnson, D., Tarjan, R.: The planar hamiltonian circuit problem is NP-complete. SIAM Journal of Computing 5, 704–714 (1976)
16. Gharan, S.O., Saberi, A., Singh, M.: A Randomized Rounding Approach to the Traveling Salesman Problem (2011) (manuscript)
17. Grigni, M., Koutsoupias, E., Papadimitriou, C.: An approximation scheme for planar graph TSP. In: Proc. 36th Annual Symposium on Foundations of Computer Science, pp. 640–645 (1995)
18. Hartvigsen, D., Li, Y.: Maximum cardinality simple 2-matchings in subcubic graphs, University of Notre Dame (2009) (manuscript)
19. Kaiser, T., Král', D., Norine, S.: Unions of perfect matchings in cubic graphs. Electronic Notes in Discrete Math. 22, 341–345 (2005)
20. Lawler, E.L., Lenstra, J.K., Rinnooy Kan, A.H.G., Shmoys, D.B.: The Traveling Salesman Problem–A Guided Tour of Combinatorial Optimization. Wiley, Chichester (1985)
21. Naddef, D., Pulleyblank, W.: Matchings in regular graphs. Discrete Math. 34, 283–291 (1981)
22. Papadimitriou, C., Yannakakis, M.: The traveling salesman problem with distances one and two. Math. Oper. Res. 18, 1–11 (1993)
23. Petersen, J.: Die Theorie der regulären graphen. Acta Math. 15, 193–220 (1891)
24. Shmoys, D., Williamson, D.: Analyzing the Held-Karp TSP bound: A monotonicity property with application. Information Processing Letters 35, 281–285 (1990)
25. Wolsey, L.: Heuristic analysis, linear programming and branch and bound. Math. Programming Study 13, 121–134 (1980)